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LETTER TO THE EDITOR

Elasticity and failure of a set of elements loaded in parallel

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Abstract. We study the elasticity and failure characteristics of a system built by 'association in parallel' of links with identical spring constant but random failure thresholds. The occurrence of the first link rupture, which is related to the theory of extreme order statistics, is contrasted with the global failure for which a central limit theorem holds. We also discuss the similarities and differences between the 'democratic' and 'hierarchical' models which differ in the mechanism by which the stress supported by links which have failed is transferred to the other stronger links. This problem gives the simplest model of elastic non-linear behaviour before global rupture occurs.

Failure is often associated with the statistics of extremes (Gumpel 1958, Galambos 1978, Jayatilaka 1979): the weakest part of a system submitted to a given load fails first and this may sometimes trigger a macroscopic failure. The paradigm of this regime is the model of links with randomly distributed failure thresholds associated in series (Jayatilaka 1979).

Most physical or mechanical systems are more complex and one must consider models where several links share the total load. Consider for example a Euclidean lattice of L^d bonds with randomly distributed failure thresholds and suppose that the stress is applied along the z direction. In this case, both parallel and series association are involved in a complex mixed way. Determining the global rupture properties constitutes an extremely difficult problem since it involves non-local long-range screening and enhancement effects as well as connectivity effects which cannot be described easily by perturbative or probabilistic approaches. The status of the Euclidean lattices which are obtained by associating bonds both in series and in parallel is not addressed here and remains a challenge (see De Arcangelis *et al* (1988) for numerical work giving some insight into this problem (Sornette 1989)).

Let us consider the following simplified version of the problem. Consider n independent parallel vertical lines with identical spring constant κ^{-1} but random failure thresholds $X_j, j = 1, \ldots, n$. Furthermore, suppose that a total stress S is applied to the system. Depending on the way in which the total stress S is shared among the n links, we obtain different problems. We will consider the simplest case of a 'democratic' distribution: we suppose that each of the n links supports a fraction S/n of the whole applied stress S. Note that the problem is posed in similar terms in the electrical or mechanical context with the correspondence (Gilabert *et al* 1987) of spring constant to link conductance, mechanical stress to electric current density, and mechanical strain to electric voltage. In the following, we use the mechanical language. This problem, to our knowledge, was first discussed by Daniels (1945). It can model a variety of systems such as cables or ropes made of numerous fibres, geological faults

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which are locked by asperity barriers sharing the total stress (Smalley *et al* 1985, Turcotte *et al* 1985), electric networks, etc.

The purpose of this letter is to discuss the mechanical and failure characteristic of the type of system built by association of links in parallel. We will see that the rupture properties of these systems are very different from those of systems of links associated in series (Coleman 1958, Galambos 1978). A second goal of this work is to improve the treatment of the failure threshold presented by Smalley *et al* (1985), which relied on an approximate hierarchical mechanism for stress transfer (see below) (see also Rosen (1964) and Suemasu (1984) for preliminary comparison between democratic and hierarchic models). This hypothesis is not in general justifiable but was made for the purpose of solving the problem. It turns out that a central limit theorem exists in the case where the stress transfer mechanism is 'democratic' and that no simplifying assumption is needed.

We find the following results.

(i) The system is elastic and reversible in the range $0 \leq S \leq S_1$ where S_1 is the stress threshold at which the first link failure occurs. S_1 is described by the theory of extreme order statistics and in general decreases to zero as n increases to infinity.

(ii) For $S_1 \leq S \leq S_n$, where S_n is the global failure threshold, the system exhibits a non-linear elastic behaviour. After deloading, the system comes back to the origin, i.e. to the unstrained state. This regime is similar to the behaviour of a brittle system with a single crack which propagates under imposed displacement. S_n can be shown to be described by a central limit theorem and scales as $S_n \sim n$.

(iii) For $S > S_n$, global failure occurs.

In the following, we summarise the central limit theorem, explore its consequences and compare them with the results obtained within the hierarchical assumption of Smalley *et al* (1985).

Let us denote the strengths of the individual links by $X_1, X_2, \ldots, X_n$, and suppose that they are independent identically distributed random variables with the cumulative probability distribution $P(X_j < x) = F(x)$. Furthermore, assume that the total load S is distributed equally on the individual links. Under a total load S, a fraction $F(S/n)$ of the threads will be submitted to more than their rated strength and will fail immediately. After this first step, the total load will be redistributed by the transfer of stress from the broken links to the other unbroken links. This transfer will in general induce secondary failures which in turn induce tertiary ruptures, and so on. The problem consists of describing this cascade of induced failure. An important question is: does this cascade stop or propagate until the whole sample is broken? The answer depends on the way the total stress is redistributed over the remaining links. One has therefore to face an intricate n-body non-linear problem. This problem has been solved by Smalley *et al* (1985) in the approximation that the stress transfer was restricted to the adjacent link inside the same cell in a hierarchical lattice. When both links of a cell have failed, the total stress on these two links was transferred to the cell belonging to the next stage of the hierarchical tree. We will compare the result of this 'hierarchical' model to the 'democratic' model which is now discussed.

Consider the other limit for the propagation of the failure among the n links or, in other words, for the successive transfer of stress: when a thread fails, the stress on the failed link is supposed to be transferred 'democratically' to the other links. This problem seems much more difficult than the hierarchical model but it turns out that it can be solved using the theory of extreme order statistics (Galambos 1978). An heuristic derivation can be also found in Coleman (1958). The clue to the solution is that, evidently, the bundle will not break under a load S if there are k links in the bundle, each of which can withstand S/k. In other words, if $X_{1;n} \leq X_{2;n} \leq \ldots \leq X_{n;n}$ are the ordered strengths of the individual links, then if we remove the first $k-1$ weakest links, the bundle will resist under a stress less than or equal to $(n-k+1)X_{k,n}$ since there remain $(n-k+1)$ links of breaking strength greater than or equal to $X_{k,n}$. Then, the strength S_n of the bundle is given by

$$S_n = \max\{(n-k+1)X_{k;n}\colon 1 \leq k \leq n\}. \tag{1}$$

We are not looking for the weakest link out of the n links (as in the case of links associated in series or for determining the occurrence of the weakest link failure) but are searching for the *strongest subgroup.* The variables $X_{k;n}$ are strongly dependent since they are ordered. The probabilistic description of S_n which is a partial sum of correlated random variables may thus be expected to be difficult. However, a very general result holds for S_n independent of the specific distribution $F(x)$. The result we will use throughout this letter has been derived by Galambos (1978) and is now summarised by the following theorem.

Theorem (Galambos 1978). Let $F(x)$ be absolutely continuous with finite second moment. Assume that $x(1-F(x))$ has a unique maximum at $x = x_0 > 0$ and let $\theta = x_0(1-F(x_0))$. If, in a neighbourhood of x_0, $F(x)$ has a positive continuous second derivative, then as $n \to +\infty$

$$\lim_{n\to\infty} P(S_n < n\theta + x\sqrt{n}) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2)\,\mathrm{d}t. \tag{2}$$

It is remarkable that the asymptotic properties of the global failure threshold of the bundle can be reduced to the so-called central limit problem. Expression (2) implies that the probability that the global failure threshold S_n be equal to S is

$$P(S_n = S) \sim (2\pi n x_0)^{-1/2} \exp[-(S-n\theta)^2/2nx_0^2]. \tag{3}$$

The density distribution of the global failure threshold is normally distributed around the maximum $S = n\theta$ with a dispersion scaling as $\sqrt{n}$. This means that the typical strength of the total system increases as $S_n \sim n$, for large n. One could argue that this result could be expected rather trivially on the basis of the independence of the random failure threshold for each parallel link. Indeed, from a naive argument, one would predict that the global failure threshold should scale as $S_n = n\langle x\rangle$ where $\langle x\rangle$ is the average one-link failure threshold. In fact, (3) shows that $S_n = n\theta$ with θ significantly smaller than $\langle x\rangle$. To illustrate this point, let us consider the usual Weibull distribution (Jayatilaka 1979) for the failure threshold of each individual link

$$F(x) = 1 - \exp[-(x/\lambda)^m]. \tag{4}$$

We easily obtain

$$x_0/\lambda = (1/m)^{1/m} \tag{5}$$

$$\theta/\lambda = (1/m)^{1/m} \exp(-1/m) \tag{6}$$

which are weakly dependent upon the order m of the Weibull distribution. For $m = 2$, one finds $\theta/\lambda = 0.429$. This value must be contrasted with $\langle x\rangle/\lambda = 0.886$ which is obtained from (4) for the same value $m = 2$. Therefore, the naive argument strongly overestimates the global failure threshold. By crudely using it, one forgets the intricate non-linear stress transfer mechanism which is very sensitive to the failure threshold fluctuations.

It is worth pointing out that the scaling of S_n is very different from the scaling of the weakest link strength

$$S_1/\lambda \sim n^{-1/m} \tag{7}$$

obtained for the Weibull distribution (4) (Sornette 1988).

The mechanical characteristics of a system under a given applied stress $S < S_n$ depend upon the history of the stress, i.e. on the number of links which have failed as the stress was increased from zero to S. It is easy to show, using the method of Galambos (1978) that, in the limit of large n, the number of links which have failed under S is

$$k(S) = nF(x(S)) \tag{8}$$

where $x(S)$ is defined by

$$S/n = x(S)(1 - F(x(S)). \tag{9}$$

The number of remaining links is therefore $n(1 - F(x(S))) = S/x(S)$.

Equations (8) and (9) can be inferred from the following reasoning. With each value of $S < S_n$, we can associate an integer $p(S)$ with $1 \leq p(S) \leq n$ such that

$$(n - p(S) + 2)X_{p-1;n} \leq S \leq (n - p(S) + 1)X_{p;n}$$

which can be rewritten as

$$[1 - (p(S) - 2)/n]X_{p-1;n} \leq S/n \leq [1 - (p(S) - 1)/n]X_{p;n}. \tag{10}$$

Here $p(S) - 2$ is the number of links which have failed under a stress less than or equal to $S/(n - p(S) + 2)$. We have also, by definition of $F(x)$,

$$(p - 2)/n \leq F(S/(n - p(S) + 2)) \leq (p - 1)/n. \tag{11}$$

Expression (11) follows from the fact that, for large n, counting the number of links with failure threshold less than $S/(n - p(S) + 2)$ amounts to computing the cumulative failure distribution $F(x)$ at $x = S/(n - p(S) + 2)$. Roughly speaking, expression (11) with (10) shows that, as n increases to infinity, S/n is better and better approximated by $x(1 - F(x))$ with x given by (9). Using the method of Galambos (1978), it is then easy to make this argument rigorous by following step by step the demonstration of (2). Note that (9) is a continuous function in the limit $n \to +\infty$. For large but finite n, $S(x)$ or $x(S)$ is a staircase with plateaux of width decreasing to zero as $n \to +\infty$. For a given S interval, the width of each plateau can be obtained from (10) as the interval in S such that (10) holds with the same integer $p(S) = p$.

Just before complete failure of the bundle, the total number of broken links, as obtained from (8) and (9), is

$$k_n = k(S_n) = nF(x_0). \tag{12}$$

Therefore, a finite fraction of the links fail before global rupture occurs. For the Weibull distribution (4), the fraction k_n/n of broken links just before global failure is

$$k_n/n = 1 - \exp(-1/m). \tag{13}$$

For $m = 2$, this gives $k_n/n = 0.393$.

For $S \leq S_n$, $x(S)$ is in the neighbourhood of x_0 and can be expressed in the form

$$x(S) - x_0 = -A(\theta - S/n)^{1/2} \tag{14}$$

where A is a coefficient which depends upon the cumulative distribution $F(x)$. For the Weibull distribution (4), $A = [x_0 \exp(1/m)/m]^{1/2}$. The number of links which have failed under the stress S is obtained from the asymptotic form of (8) with (14):

$$k(S)/n = F(x_0) - B(\theta - S/n)^{1/2} \tag{15}$$

with $B = (mx_0)^{-1/2} \exp(-1/2m)$ for the Weibull distribution (4). As depicted in figure 1, expression (15) shows a very rapid increase of the number of broken links as $S \to S_n$ for which $k(S)$ tends to $n(F(x_0)$ with a square-root singularity.

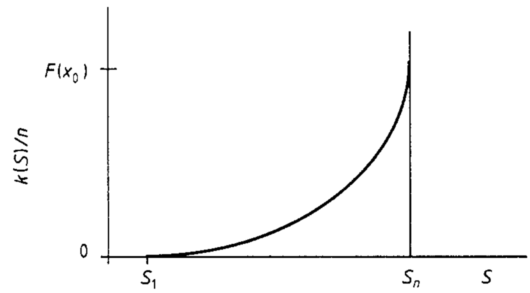

Figure 1. Fraction of links $k(S)/n$ which have failed under a stress less than or equal to S. Note the zero slope of $k(S)/n$ at $S = S_1$ and the infinite slope at $S = S_n$.

Note that under the limiting stress S_n, a fraction $F(S_n/n = \theta)$ of the links immediately breaks down and, due to stress transfer, other links break in cascade until a total fraction $F(x_0)$ of broken links is reached. From (5) and (6), one has $F(x_0) = 1 - \exp(-1/m)$ and $F(\theta) = 1 - \exp(-e^{-1}/m)$. For $m = 2$, a fraction $F(\theta) = 0.168$ of links fail immediately under S_n but the cascade of stress transfer to the other links results in a total fraction $F(x_0) = 0.393$ of broken links.

We can also predict the strain(ε)-stress(σ) characteristic of the bundle of threads. Suppose that each individual link has a linear characteristic $\varepsilon = \kappa\sigma$ until its failure (brittle systems).

For $S \leqslant S_1$ ($\sim \lambda n^{-1/m}$ for the Weibull distribution (4)), all links are intact and the system has a linear stress-strain characteristic with slope κ^{-1}.

For $S_1 \leqslant S \leqslant S_n$, some links have failed and the system is elastic non-linear as we now show. From (8), $n(1 - F(x(S)))$ links support the total stress S. Therefore, the stress per remaining link is given by

$$\sigma = S/n(1 - F(x(S))) = x(S) \tag{16}$$

where we have used (9). Note that (16) gives an intuitive physical meaning to $x(S)$. To σ there corresponds a strain per link (equal to the strain of the total bundle of links associated in parallel) equal to

$$\varepsilon = \kappa x(S). \tag{17}$$

From (14) with (17), we predict a (ε, σ) characteristic which becomes flat with zero slope as one approaches the global failure threshold $S \to S_n$ as shown in figure 2. The apparent elastic modulus therefore decreases as S increases. Intuitively, this non-linear behaviour stems from the fact that as S tends to S_n, more and more links fail and as a consequence the whole stress is supported by fewer and fewer links. This non-linear transfer of the stress to fewer and fewer links is the basic mechanism for the non-linearity of the (ε, σ) characteristic. This argument is reminiscent (although opposed in the conclusion) to the apparent increase of the elastic modulus of a granular array of inhomogeneous grains under pressure (Roux *et al* 1987). In this case, increasing the

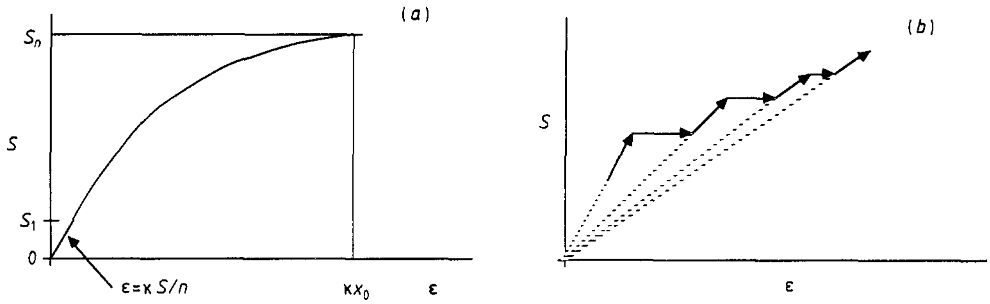

Figure 2. (*a*) Stress-strain characteristic of the bundle of n links with identical spring constant κ^{-1} but random failure thresholds. (*b*) Magnified view of the apparent smooth characteristic represented in figure 2(*a*). Note that the curve is obtained under an applied stress. The horizontal segments correspond to the rupture of a single bond. Note that each linear tilted portion of the characteristic, which represents a period with no failure, goes through the origin.

applied stress results in an increase in the number of active contacts and thus to an increase of the apparent elastic modulus.

We note that this non-linear behaviour of the (ε, σ) curve is characteristic of an irreversible system. This corresponds indeed to the finite and irreversible deterioration of the bundle as S increases towards S_n.

Let us now summarise the above results and compare the 'democratic' model and the 'hierarchical' model studied by Smalley *et al* (1985) by pointing out the analogues and differences.

	Hierarchical model	Democratic model
Nature of the global failure collapse	Continuous or 'critical' transition	Abrupt or 'first-order' transition
Threshold value F^* of the failure distribution at failure collapse for $m=2$: this is also the fraction of links with strength less than S_n	0.206	$F^*=F(\theta)=0.168$
Number $k(S)$, of broken links as $F\to F^*$	$k(S)\sim(F^*-F)^{-\nu}\times f(n(F^*-F)^{\nu})$ with $f(x)=x$ for $x\to 0$; therefore, $k(S)=n$ for $F=F^*$, $\nu=1.439$ $(d=1)$ $\nu=0.808$ $(d=2)$	$k(S)\to nF(x_0)$ (see equation (7)) $=0.393n$ for $m=2$
Susceptibility of the failure precursors which are supposed to be proportional to $k(S)$	Power-law divergence	Finite but rapid increase with a square-root singularity near F^*

In spite of the difference in natures of the collapse transitions, the two models present similar behaviours: (i) there is a rapid increase of the number of broken bonds as the collapse transition is approached; (ii) the threshold value $F^* = 0.168$ of the failure distribution at failure is much smaller than the value $1 - \exp[-(0.886)^2] \approx 0.544$ which corresponds to the mean strength of the links $\langle x \rangle = 0.886\lambda$ (values given for the Weibull distribution with $m = 2$). This means that below the collapse load threshold, very few precursory failures have occurred. This is in agreement with observation in the geophysical context (Turcotte *et al* 1985).

The problem of the rupture of a system built by 'association of links in parallel' can be analysed within the theory of extreme order statistics. The main surprising ingredient underlying the whole discussion presented in this letter is the validity of a central limit theorem for the global failure threshold. This approach is instructive since it gives another point of view and a powerful analytical method as an alternative to the naive intuitive non-linear transfer cascade approach which has been solved only for hierarchical systems (Smalley *et al* 1985). In this letter, we have used these results in order to extract statistical quantities such as the number of broken bonds under a given load as well as the form of the stress-strain non-linear characteristic. The 'democratic' model is also instructive in comparison with the 'hierarchical' model.

In order to be able to distinguish between the 'democratic' and the 'hierarchical' models, for example in the geological context of faults stabilised by asperities, one will need precise data on the behaviour of precursors in order to distinguish between (15) and a critical singularity of the form $k(S) \sim (F^* - F)^{-\nu}$ derived in Smalley *et al* (1985). Note that the continuous nature of the collapse transition in the hierarchical model is deeply related to the built-in self-similar hierarchical structure and may not be relevant in most real situations. The comparison between the two models cautions against the validity of the results obtained with simplifications such as the hierarchical stress transfer mechanism. Finally, this problem gives the simplest model of non-linear behaviour before global rupture occurs.

In a forthcoming work (Sornette 1989) which can be considered as a sequel to the present letter, exact extended results on a class of hierarchical systems are discussed within a real space renormalisation group (RG). An approximate RG is also proposed to treat the case of two- and three-dimensional Euclidean systems.

Future extensions also concern models of composite materials made of parallel fibres immersed in a matrix along the lines of Harlow and Phoenix (1981, 1982) and Kuo and Phoenix (1987). Here, the analysis is also based on the 'chain-of-bundles' model but with local load sharing assumed for the non-failed fibres in a bundle. These authors obtain bounds for the probability distribution of strength which rely on the occurrence of k or more adjacent broken fibres in a bundle. It would be desirable to have the analogue of a central limit theorem or a renormalisation group for this problem since it is the one which is present in practice in fibre composites. We hope to return to this question in a future publication.

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